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Complete Convergence for Weighted Sums of Arrays of Rowwise $\tilde{\rho}$-mixing Random Variables

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Abstract

In this paper we obtain some new results on complete convergence for weighted sums of arrays of rowwise $\tilde{\rho}$-mixing random variables. Our results improve and extend the some results established for sequences of independent random variables.

Keywords: complete convergence, $\tilde{\rho}$-mixing random variables, slowly varying function, weighted sums.

1. Introduction

The concept of complete convergence was introduced by Hsu and Robbins in [1] as follows: A sequence of random variables $\{X_n, n \geq 1\}$ are said to converge completely to a constant C if $\sum_{n=1}^{\infty} P(|X_n - C| > \varepsilon) < \infty$ for all $\varepsilon > 0$. From then on, many authors have devoted their study to complete convergence.

Recently, Sung [2] proved the following two results. In Theorems A and B we assume that $\{X_n, n \geq 1\}$ is a sequence of zero-mean independent random variables stochastically dominated by a random variable X, that is, $P(|X_n| > x) \leq CP(|X| > x)$ for all $x > 0$ and all $n \geq 1$ and some positive constant C. Moreover, $\{a_{ni}, i \geq 1, n \geq 1\}$ is an array of real numbers satisfying such that $\sup_{n\geq 1, i\geq 1} |a_{ni}| < \infty$ and $\sum_{i=1}^{\infty} a_{ni} X_i$ is finite almost surely for all $n \geq 1$. Finally, let $t \geq -1, -\infty < \beta < \infty, p > 0$ be constants such that $\gamma = p(t + \beta + 1) > 0$.

Theorem A. *Assume that* $E|X|^{\gamma} < \infty$ *and*

$$\sum_{i=1}^{\infty} |a_{ni}|^{\alpha} = O(n^{\beta}) \ \textit{for some } \alpha < \gamma. \tag{1}$$

(i) If $1 \leq \gamma < 2$, or

(ii) if $\gamma \geq 2$, and

$$\sum_{i=1}^{\infty} a_{ni}^2 = O(n^q) \ \textit{for some } q < 2/p, \tag{2}$$

then

$$\sum_{n=1}^{\infty} n^t P(n^{-1/p} |\sum_{i=1}^{\infty} a_{ni} X_i| > \varepsilon) < \infty \ \textit{for all } \varepsilon > 0. \tag{3}$$

Theorem B. *Assume that* $E|X|^{\gamma} \log|X| < \infty$ *and*

$$\sum_{i=1}^{\infty} |a_{ni}|^{\gamma} = O(n^{\beta}) \ \textit{for all } n \geq 1. \tag{4}$$

(i) $\textit{If } 1 \leq \gamma < 2 \textit{ or}$

(ii) If $\gamma \geq 2$ *and* $\sum_{i=1}^{\infty} a_{ni}^2 = O(n^q)$ for some $q < 2/p$,

then (3) holds.

Let Z be the set of integers and $\{a_{nk}, n \geq 1, k \in Z\}$ be an array of constants. Denote

$$N(n, m+1) = \#\{k \in Z : |a_{nk}| \geq (m+1)^{-1/p}\}, p \geq 2, n \geq 1, m \geq 1,$$

where the symbol $\#A$ stands for the number of elements in the set A. For two sequences of real numbers $\{a_m, m \geq 1\}$ and $\{b_m, m \geq 1\}$, we write $a_m \approx b_m$ as $m \to \infty$, if $a_m = O(b_m)$ and vise versa $b_m = O(a_m)$ as $m \to \infty$.

Wang et al. [3] proved the following result:

Theorem C. *Let* $r > 1$ *and* $\{X_i, i \in Z\}$ *be a sequence of i.i.d. random variables and let* $\{a_{ni}, n \geq 1, i \in Z\}$ *for be an array of constants.*

(I) *If* $p > 2$ *and*

$$N(n, m+1) \approx m^{q(r-1)/p}, n \geq 1, \text{ as } m \to \infty, \text{ when } 2 \leq q < p, \qquad (5)$$

$$EX = 0, \text{ when } 1 \leq q(r-1), \qquad (6)$$

$$\sum_{i \in Z} a_{ni}^2 = O(n^{\delta}), n \to \infty, \text{ when } 2 \leq q(r-1), \text{ for some } 0 < \delta < 2/p, \quad (7)$$

then the following statements are equivalent:

$$\text{(i)} E|X|^{p(r-1)} < \infty;$$

$$\text{(ii)} \sum_{i=1}^{\infty} 2^{i(r-1)} \max_{2^{i-1} \leq n < 2^i} P\left(n^{-1/p} \left| \sum_{k \in Z} a_{nk} X_k \right| > \varepsilon \right) < \infty \text{ for all } \varepsilon > 0.$$

(II) *If* $p = q = 2$ *and*

$$N(n, m+1) \approx m^{(r-1)}, n \geq 1, \text{ as } m \to \infty,$$

$$EX = 0, \text{ when } 1 \leq 2(r-1),$$

$$\sum_{i\in Z} |a_{ni}|^{2(r-1)} = O(1), n \to \infty,$$

then the following statements are equivalent:

$$\text{(i)} E|X|^{2(r-1)} \log(1+|X|) < \infty;$$

$$\text{(ii)} \sum_{i=1}^{\infty} 2^{i(r-1)} \max_{2^{i-1}\le n<2^i} P\left(n^{-1/2} \left| \sum_{k\in Z} a_{nk} X_k \right| > \varepsilon \right) < \infty, \text{ for all } \varepsilon > 0.$$

The main purpose of this paper is to generalize the above mentioned results for $\tilde{\rho}$-mixing random variables (see the definition below). Theorem A and Theorem B and the sufficient part of Theorem C are extended and improved for $\tilde{\rho}$-mixing case.

Let $\{\Omega, \Im, P\}$ be a probability space. In the following, all random variables are assumed to be defined on $\{\Omega, \Im, P\}$. For a sequence of random variables $\{X_n, n \ge 1\}$ we denote $\Im_S = \sigma(X_n : n \in S \subset N)$. Given two σ-subalgebras $\Im_1, \Im_2 \subset \Im$, denote

$$\rho(\Im_1, \Im_2) = \sup\{|\operatorname{corr}(\zeta, \eta)|, \zeta \in L_2(\Im_1), \eta \in L_2(\Im_2)\},$$

where the correlation coefficient is defined in usual way

$$\operatorname{corr}(\zeta, \eta) = \frac{E(\zeta\eta) - E\zeta E\eta}{\sqrt{\operatorname{Var}(\zeta)\operatorname{Var}(\eta)}}$$

and by $L_2(\Im)$ we denote the space of all $\Im$-measurable random variables ζ such that $E(\zeta^2) < \infty$.

Stein [4] introduced the following coefficients of dependence (with slightly different notations):

$$\tilde{\rho}(k) = \sup\{\rho(\Im_S, \Im_T) : \text{all finite subsets } S, T \subset N \text{ such that } \operatorname{dist}(S,T) \ge k\},$$

$k \ge 0$. Obviously, $0 \le \tilde{\rho}(k+1) \le \tilde{\rho}(k) \le 1, k \ge 0,$ and $\tilde{\rho}(0) = 1$.

Definition. A sequence of random variables $\{X_n, n \geq 1\}$ are said to be a $\tilde{\rho}$ *-mixing sequence* if there exists $k \in N$ such that $\tilde{\rho}(k) < 1$. An array of random variables $\{X_{nk}, k \geq 1, n \geq 1\}$ are said to be an array of rowwise $\tilde{\rho}$ -mixing random variables, if, for every positive integer n the sequence of random variables $\{X_{nk}, k \geq 1\}$ is a $\tilde{\rho}$ -mixing sequence.

For fixed n -th row of an array of rowwise $\tilde{\rho}$ -mixing random variables $\{X_{nk}, n \geq 1, k \geq 1\}$ we denote the coefficients of dependence of the sequence $\{X_{nk}, k \geq 1\}$ as $\tilde{\rho}_n(\cdot)$ for every $n \geq 1$.

The notion of $\tilde{\rho}$ -mixing assumption is similar to ρ -mixing, but they are quite different from each other. A number of publications are devoted to $\tilde{\rho}$ -mixing sequence. We refer to Bradley [5,6] for the central limit theorem, Bryc and Smolenski [7] for moment inequalities and almost sure convergence, Shanchao [8] for moment inequalities and strong law of large numbers, Gut and Peligrad [9], Wu [10,11], and Shixin [12] for almost sure convergence, Utev and Peligrad [13] for maximal inequalities and the invariance principle, Dehua and Shixin [14,15] for complete convergence, Dehua and Shixin [16] for Hájeck-Rènyi inequality and strong law of large numbers among many others.

Recall that a measurable function h is said to be *slowly varying* if for each $\lambda > 0$

$$\lim_{x \to \infty} \frac{h(\lambda x)}{h(x)} = 1.$$

We refer to Seneta [17] for other equivalent definitions and for detailed and comprehensive study of properties of such functions.

Throughout this paper, we assume that $\sum_{i=1}^{\infty} a_{ni} X_{ni}$ is finite almost surely, C is a positive constant which may vary from one place to another, the symbol $[x]$ denotes the greatest integer less than x, and the symbol $\lfloor x \rfloor$ denotes the least integer more than x.

2. Lemmata

In order to prove our main result, we need the following lemmas. The proof of the first lemma could be found in Utev and Peligrad [13].

Lemma 1. *For a positive integer* J *and* $0 \le r < 1$ *and* $u \ge 2$, *there exists a positive constant* $C = C(u, J, r)$ *such that if* $\{X_n, n \ge 1\}$ *is a sequence of random variables with* $\tilde{\rho}(J) \le r, EX_k = 0$, *and* $E|X_k|^u < \infty$ *for every* $k \ge 1$, *then for all* $n \ge 1$,

$$E \max_{1 \le i \le n} \left| \sum_{k=1}^{i} X_k \right|^u \le C \left\{ \sum_{k=1}^{n} E|X_k|^u + \left(\sum_{k=1}^{n} EX_k^2 \right)^{u/2} \right\}.$$

The second lemma is well known and we do not present the proof.

Lemma 2. *Let* $\{X_{nk}, n \ge 1, k \ge 1\}$ *be an array of random variables stochastically dominated by a random variable X, then there exists a constant* D *such that for all* $u > 0$ *and* $x > 0$,

(i) $E|X_{nk}|^u I(|X_{nk}| \le x) \le D\{E|X|^u I(|X| \le x) + x^u P(|X| > x)\}$,

(ii) $E|X_{nk}|^u I(|X_{nk}| > x) \le DE|X|^u I(|X| > x)$.

The proof of the last lemma could be found in Bai and Su [18]

Lemma 3. *Let* $h(x) > 0$ *be a slowly varying function as* $x \to +\infty$, *then*

(i) $\lim_{k \to +\infty} \sup_{2^k \le x \le 2^{k+1}} \frac{h(x)}{h(2^k)} = 1,$

$$\lim_{x \to +\infty} x^{\delta} h(x) = +\infty, \quad \lim_{x \to \infty} x^{-\delta} h(x) = 0, \text{ for all } \delta > 0.$$

(ii) For all $\delta > 0, \eta > 0$, and all positive integers k

$$C \cdot 2^{k\delta} h(2^k \cdot \eta) \le \sum_{j=1}^{k} 2^{j\delta} h(2^j \cdot \eta) \le C \cdot 2^{k\delta} h(2^k \cdot \eta).$$

(iii) For all $\delta < 0, \eta > 0$ all positive integers k

$$C \cdot 2^{k\delta} h(2^k \cdot \eta) \leq \sum_{j=k}^{\infty} 2^{j\delta} h(2^j \cdot \eta) \leq C \cdot 2^{k\delta} h(2^k \cdot \eta).$$

3. Main Results and Proofs

With the preliminaries accounted for, we can now formulate and prove main results of this paper.

Theorem 1. *Let* $p > 0, t, \beta$ *be constants such that* $t + \beta > -1$*,* $h(x) > 0$ *be a slowly varying function,* $\{X_{nk}, n \geq 1, k \geq 1\}$ *be an array of zero-mean rowwise* $\tilde{\rho}$ *-mixing random variables stochastically dominated by a random variable* X *, and* $\{a_{ni}, i \geq 1, n \geq 1\}$ *be an array of constants satisfying (1). Assume that*

$$\overline{\lim_{k\to\infty}} \sup_n \tilde{\rho}_n(k) < 1 \text{ and } E|X|^\gamma h(|X|^p) < \infty,$$

where $\gamma = p(t + \beta + 1) > 0$.

If $t = -1$ *we additionally assume that* $E|X|^\gamma < \infty$.

(i) If $\gamma = 1$*, and* $E|X| < \infty$*, then*

$$\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} P\left(n^{-1/p} \left| \sum_{i=1}^{\infty} a_{ni} X_{ni} \right| > \varepsilon \right) < \infty \text{ for all } \varepsilon > 0, \quad (8)$$

moreover

$$\sum_{n=1}^{\infty} n^t h(n) P\left(n^{-1/p} \left| \sum_{i=1}^{\infty} a_{ni} X_{ni} \right| > \varepsilon \right) < \infty \text{ for all } \varepsilon > 0. \quad (9)$$

(ii) If $1 < \gamma < 2$ *, then (8) and (9) hold.*

(iii) If $\gamma = 2, \{a_{ni}, n \geq 1, i \geq 1\}$ *satisfies (2), and* $E|X|^2 < \infty$*, then (8) and (9) hold.*

(iv) If $\gamma \geq 2$ *and* $\{a_{ni}, n \geq 1, i \geq 1\}$ *satisfies (2), then (8) and (9) hold.*

Proof. First of all we note that it is enough to show that (8) holds. Really, by Lemma 3 we have

$$\sum_{n=1}^{\infty} n^{t} h(n) P\left(n^{-1/p} \left|\sum_{i=1}^{\infty} a_{ni} X_{ni}\right| > \varepsilon \right)$$

$$= \sum_{j=0}^{\infty} \sum_{2^{j} \leq n < 2^{j+1}} n^{t} h(n) P\left(n^{-1/p} \left|\sum_{i=1}^{\infty} a_{ni} X_{ni}\right| > \varepsilon \right)$$

$$\leq C + C\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^{j}) \max_{2^{j} \leq n < 2^{j+1}} P\left(n^{-1/p} \left|\sum_{i=1}^{\infty} a_{ni} X_{ni}\right| > \varepsilon \right) < \infty,$$

therefore, (9) holds.

If $t < -1$, then by Lemma 3 (i) we obtain that (8) holds. Thus, we assume that $t \geq -1$. Since $\sum_{i=1}^{\infty} a_{ni} X_{ni}$ is finite almost surely for each $n \geq 1$, there exists positive integer k_n such that

$$P(n^{-1/p} \left|\sum_{i=k_n+1}^{\infty} a_{ni} X_{ni}\right| > \varepsilon/2) < n^{-t-2)}, \text{ for all } n \geq 1.$$

By Lemma 3 (iii), in order to prove (8), it is enough to show that

$$\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^{j}) \max_{2^{j} \leq n < 2^{j+1}} P(n^{-1/p} \left|\sum_{i=1}^{k_n} a_{ni} X_{ni}\right| > \varepsilon / 2) < \infty. \qquad (10)$$

Without loss of generality, we assume that $a_{ni} > 0$ for all $n \geq 1, i \geq 1, \sup_{i \geq 1, n \geq 1} a_{ni} = 1$, and $\sum_{i=1}^{\infty} a_{ni}^{\alpha} \leq n^{\beta}$. Thus, for any $\theta \geq 0$, we have

$$\sum_{i=1}^{\infty} a_{ni}^{\alpha+\theta} \leq n^{\beta}. \qquad (11)$$

For $i \geq 1, n \geq 1$ we define

$$U_{ni} = X_{ni} I(|a_{ni} X_{ni}| \leq n^{1/p}), V_{ni} = X_{ni} I(|a_{ni} X_{ni}| > n^{1/p}).$$

Since $EX_{ni} = 0$, we obtain

$$\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} P(n^{-1/p} |\sum_{i=1}^{k_n} a_{ni} X_{ni}| > \varepsilon/2)$$

$$\le \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} P(n^{-1/p} |\sum_{i=1}^{k_n} a_{ni} (U_{ni} - EU_{ni})| > \varepsilon/4)$$

$$+ \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} P(n^{-1/p} |\sum_{i=1}^{k_n} a_{ni} (V_{ni} - EV_{ni})| > \varepsilon/4)$$

$$\stackrel{Def}{=} J_1 + J_2. \tag{12}$$

We estimate each term J_1 and J_2 separately.

For J_2, we first prove that

$$n^{-1/p} |\sum_{i=1}^{k_n} a_{ni} EV_{ni}| \to 0 \text{ as } n \to \infty. \tag{13}$$

If $\gamma = 1$, since $E|X| < \infty$, by Lemma 2 and (11), we have

$$n^{-1/p} |\sum_{i=1}^{k_n} a_{ni} EV_{ni}| \le n^{-1/p+\beta} E|X| I(|X| > n^{1/p})$$

$$\le n^{-(t+1)} E|X| I(|X| > n^{1/p}) \to 0 \text{ as } n \to \infty.$$

If $\gamma > 1, t > -1$, select η such that $\max\{p\beta, \alpha, 1\} < \eta < \gamma$. Since $E|X|^{\gamma} h(|X|^p) < \infty$, then by Lemma 3 (i), we have $E|X|^{\eta} < \infty$. Therefore, by Lemma 2 and (11), we obtain

$$n^{-1/p} |\sum_{i=1}^{k_n} a_{ni} EV_{ni}| \le n^{-1/p} \sum_{i=1}^{\infty} n^{-(\eta-1)/p} E|a_{ni} X_{ni}|^{\eta} I(|a_{ni} X_{ni}| > n^{1/p})$$

$$\le n^{-\eta/p+\beta} E|X|^{\eta} I(|X| > n^{1/p})$$

$$= n^{(p\beta-\eta)/p} E|X|^{\eta} I(|X| > n^{1/p}) \to 0 \text{ as } n \to \infty.$$

If $\gamma > 1, t = -1$, since $E|X|^{\gamma} < \infty$ we obtain

$$n^{-1/p} \left|\sum_{i=1}^{k_n} a_{ni} EV_{ni}\right| \le n^{-1/p} \sum_{i=1}^{\infty} n^{-(\gamma-1)/p} E|a_{ni}X_{ni}|^{\gamma} I(|a_{ni}X_{ni}| > n^{1/p})$$

$$\le n^{-\gamma/p+\beta} E|X|^{\gamma} I(X| > n^{1/p})$$

$$= E|X|^{\gamma} I(X| > n^{1/p}) \to 0 \text{ as } n \to \infty.$$

Thus, (13) holds. Hence, there exists n large enough such that

$$n^{-1/p} \left|\sum_{i=1}^{k_n} a_{ni} EV_{ni}\right| < \varepsilon/8. \tag{14}$$

Select $\delta > 0$ such that $\gamma - \delta > 0$ and $\gamma - \delta > \alpha$, by (14), (11), Lemma 2 (ii) and Lemma 3 (ii),we have

$$J_2 \le C + \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} P\left(n^{-1/p} \left|\sum_{i=1}^{k_n} a_{ni} V_{ni}\right| > \varepsilon/8\right)$$

$$\le C + \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} \sum_{i=1}^{k_n} P(|a_{ni}X_{ni}| > n^{1/p})$$

$$\le C + \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} \sum_{i=1}^{\infty} n^{-(\gamma-\delta)/p} E|a_{ni}X_{ni}|^{\gamma-\delta} I(|a_{ni}X_{ni}| > n^{1/p})$$

$$\le C + \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) 2^{-j(\gamma-\delta)/p + j\beta} E|X|^{\gamma-\delta} I(|X| > 2^{j/p})$$

$$= C + \sum_{j=0}^{\infty} 2^{j\delta/p} h(2^j) E|X|^{\gamma-\delta} I(|X| > 2^{j/p})$$

$$= C + \sum_{j=0}^{\infty} 2^{j\delta/p} h(2^j) \sum_{i=j}^{\infty} E|X|^{\gamma-\delta} I(2^{i/p} < |X| \le 2^{(i+1)/p})$$

$$= C + \sum_{i=0}^{\infty} E|X|^{\gamma-\delta} I(2^{i/p} < |X| \le 2^{(i+1)/p}) \sum_{j=0}^{i} 2^{j\delta/p} h(2^j)$$

$$\le C + C\sum_{i=0}^{\infty} h(2^i) E|X|^{\gamma} I(2^{i/p} < |X| \le 2^{(i+1)/p})$$

$$\le C + CE|X|^{\gamma} h(|X|^p) < \infty. \qquad (15)$$

In order to estimate J_1, we first note that obviously for every positive integer n, $\{U_{ni} - EU_{ni}, 1 \le i \le k_n\}$ is a sequence of zero-mean $\tilde{\rho}$-mixing random variables with the mixing coefficient not greater than $\tilde{\rho}_n(\cdot)$.

Fix any $v \ge 2$ and $v > \gamma$ (the value of v will be specified later). By Markov's inequality, Lemma 1, and C_r-inequality, , we have

$$J_1 \le C\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} n^{-v/p} \left\{ \sum_{i=1}^{k_n} E|a_{ni}U_{ni}|^v + \left(\sum_{i=1}^{k_n} E|a_{ni}U_{ni}|^2\right)^{v/2} \right\}$$

$$\le C\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} n^{-v/p} \left\{ \sum_{i=1}^{\infty} E|a_{ni}U_{ni}|^v + \left(\sum_{i=1}^{\infty} E|a_{ni}U_{ni}|^2\right)^{v/2} \right\}$$

$$\overset{Def}{=} J_3 + J_4. \qquad (16)$$

Let $I_{nk} = \{i : (k+1)^{-1/p} < |a_{ni}| \le k^{-1/p}\}, k \ge 1, n \ge 1$, then $\cup_{k=1}^{\infty} I_{nk} = \mathbb{N}$ for all $n \ge 1$. Since $v > \gamma > \alpha$, we have $k^{(v-\alpha)/p} > j^{(v-\alpha)/p}$ for all $k > j, j, k \ge 1$.

For $\alpha > 0$

$$n^{\beta} \geq \sum_{i=1}^{\infty} |a_{ni}|^{\alpha} = \sum_{k=1}^{\infty} \sum_{i \in I_{nk}} |a_{ni}|^{\alpha} \geq \sum_{k=1}^{\infty} (\# I_{nk})(k+1)^{-\alpha/p}$$

$$\geq \sum_{k=j}^{\infty} (\# I_{nk})(k+1)^{-v/p}(j+1)^{(v-\alpha)/p}$$

$$> 2^{-\alpha/p} \sum_{k=j}^{\infty} (\# I_{nk}) k^{-v/p} j^{(v-\alpha)/p}.$$

For $\alpha < 0$, we also have

$$n^{\beta} \geq \sum_{i=1}^{\infty} |a_{ni}|^{\alpha} = \sum_{k=1}^{\infty} \sum_{i \in I_{nk}} |a_{ni}|^{\alpha} \geq \sum_{k=1}^{\infty} (\# I_{nk}) k^{-\alpha/p}$$

$$> \sum_{k=j}^{\infty} (\# I_{nk}) k^{-v/p} j^{(v-\alpha)/p}.$$

Therefore,

$$\sum_{k=j}^{\infty} (\# I_{nk}) k^{-v/p} \leq C n^{\beta} j^{-(v-\alpha)/p} \quad \textit{for all} \;\; j \geq 1. \tag{17}$$

By the same way as we proved (15) and by Lemma 2(i), we have

$$J_3 \leq C \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} n^{-\frac{v}{p}} \sum_{i=1}^{\infty} \left(n^{\frac{v}{p}} P(|a_{ni} X| > n^{1/p}) + E|a_{ni} X|^{v} I(|a_{ni} X| \leq n^{1/p}) \right)$$

$$\leq C \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} \sum_{i=1}^{\infty} P(|a_{ni} X| > n^{1/p})$$

$$+ C \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} \sum_{k=1}^{\infty} (\# I_{nk})(nk)^{-v/p} E|X|^{v} I(|X| < (n(k+1))^{1/p})$$

$$= C + C\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} \sum_{k=1}^{\infty} (\# I_{nk})(nk)^{\frac{v}{p}} \sum_{i=1}^{n(k+1)} E|X|^v I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$= C + C\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} \sum_{k=1}^{\infty} (\# I_{nk})(nk)^{\frac{v}{p}} \sum_{i=1}^{2n} E|X|^v I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$+ C\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} \sum_{k=2}^{\infty} (\# I_{nk})(nk)^{\frac{v}{p}} \sum_{i=2n+1}^{n(k+1)} E|X|^v I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$\overset{Def}{=} C + J_5 + J_6. \qquad (18)$$

Since $v > \gamma$, we have that $(\gamma - v)/p < 0$. Then by (17) and Lemma 3

$$J_5 \le C\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} n^{-v/p+\beta} \sum_{i=1}^{2n} E|X|^v I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$\le C\sum_{j=0}^{\infty} 2^{j(\gamma - v)/p} h(2^j) \sum_{i=1}^{4} E|X|^v I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$+ C\sum_{j=1}^{\infty} 2^{j(\gamma - v)/p} h(2^j) \sum_{i=5}^{2^{j+2}} E|X|^v I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$\le C + C\sum_{i=5}^{\infty} E|X|^v I((i-1)^{1/p} \le |X| < i^{1/p}) \sum_{j=\lfloor \log_2 i \rfloor - 2}^{\infty} 2^{j(\gamma - v)/p} h(2^j)$$

$$\le C + C\sum_{i=5}^{\infty} i^{(\gamma - v)/p} h(i) E|X|^v I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$\le C + CE|X|^{\gamma} h(|X|^p) < \infty. \qquad (19)$$

Next,

$$J_6 \le C\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} \sum_{i=2n+1}^{\infty} \sum_{k=[\frac{i}{n}-1]}^{\infty} (\# I_{nk})(nk)^{-v/p} E|X|^v I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$\le C\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} \sum_{i=2n+1}^{\infty} n^{-v/p+\beta} (\frac{i}{n})^{-(v-\alpha)/p} E|X|^v I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$\le C\sum_{i=3}^{\infty} i^{-(v-\alpha)/p} E|X|^v I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$+ C\sum_{j=1}^{\infty} 2^{j(t+1)} h(2^j) \sum_{i=2^j}^{\infty} 2^{j(\beta-\alpha/p)} i^{-(v-\alpha)/p} E|X|^v I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$\le CE|X|^{\alpha} + C\sum_{i=2}^{\infty} i^{-(v-\alpha)/p} E|X|^v I((i-1)^{1/p} \le |X| < i^{1/p}) \sum_{j=1}^{[\log_2 i]} 2^{j(\gamma-\alpha)/p} h(2^j)$$

$$\le C + C\sum_{i=2}^{\infty} i^{-(v-\gamma)/p} h(i) E|X|^v I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$\le C + CE|X|^{\gamma} h(|X|^p) < \infty. \quad (20)$$

Therefore, from (18), (19) , and (20) we have that $J_3 < \infty$ for $\gamma \ge 1$.

For J_4, if $\gamma \ge 2$ by (2) we have

$$\sum_{i=1}^{\infty} E|a_{ni} U_{ni}|^2 \le C\sum_{i=1}^{\infty} E|a_{ni} X|^2 \le Cn^q. \quad (21)$$

Since $q < 2/p$, we can chose v large enough such that $(t+1)+v(q/2-1/p) < 0$. By Lemma 3 (iii) we obtain

$$J_4 \le C\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} n^{-v/p} n^{vq/2} \le C\sum_{j=0}^{\infty} 2^{j\{(t+1)+v(q/2-1/p)\}} h(2^j) < \infty. \quad (22)$$

If $1 \le \gamma < 2$, let $v = 2$, then $J_4 = J_3 < \infty$. Therefore $J_1 < \infty$ for $\gamma \ge 1$. By (12), (10) holds.

Remark 1. (i) If there exists a positive constant $M > 0$ such that $h(x) \geq M$ for sufficiently large x, then the assumption $E|X|^{p(t+\beta+1)} h(|X|^p) < \infty$ implies that $E|X|^{p(t+\beta+1)} < \infty$.

(ii) Let $h(x) = 1, X_{ni} = X_i$, for all $i \geq 1, n \geq 1$, and $\{X_i, i \geq 1\}$ be a sequence of independent random variables. Then Theorem A follows from Theorem 1, since independent random variables are a special case of $\tilde{\rho}$ -mixing random variables.

(iii) Let $\beta = 0, t = r - 2$, and $h(x) = 1$. If condition (5) holds, then (1) holds according to (2.11) of Wang et al. [2], with $\alpha = \tilde{q}\ (r-1), \gamma = p(r-1), 2 \leq q < \tilde{q} < p$. When $0 < q(r-1) < 2$, by (2.11) of Wang et al. [2], we have that $\sum_{i \in Z} a_{ni}^2 = O(1)$. Therefore, if (5) and (7) hold, we have $\sum_{i \in Z} a_{ni}^2 = O(n^{\delta})$, for $0 < \delta < 2/p$. Thus Theorem 1 extends and improves the sufficient part of Theorem C (I) for the case of $\tilde{\rho}$ -mixing random variables.

If condition (1) on the weights is replaced by a weaker condition (4), we obtain the following theorem.

Theorem 2. *Let* $\{X_{nk}, k \geq 1, n \geq 1\}$ *be an array of zero-mean rowwise* $\tilde{\rho}$ *-mixing random variables stochastically dominated by a random variable* X. *Assume that* $\overline{\lim}_{k \to \infty} \sup_n \tilde{\rho}_n(k) < 1$ *and* $E|X|^{\gamma} \log|X| < \infty$, *where* $\gamma = p(t+\beta+1) > 0$ *and* $p > 0$. *Let* $\{a_{ni}, i \geq 1, n \geq 1\}$ *be an array of real numbers satisfying (4).*

(i) *If* $1 \leq \gamma < 2$, *then*

$$\sum_{j=0}^{\infty} 2^{j(t+1)} \max_{2^j \leq n < 2^{j+1}} P\left(n^{-1/p} \left| \sum_{i=1}^{\infty} a_{ni} X_{ni} \right| > \varepsilon \right) < \infty \ \ for\ all\ \varepsilon > 0, \qquad (23)$$

moreover

$$\sum_{n=1}^{\infty} n^t P\left(n^{-1/p} \left| \sum_{i=1}^{\infty} a_{ni} X_{ni} \right| > \varepsilon \right) < \infty \ \ for\ all\ \varepsilon > 0. \qquad (24)$$

(ii) If $\gamma \geq 2$ *and* $\{a_{ni}, i \geq 1, n \geq 1\}$ *satisfies (2), then (23) and (24) hold.*

Proof. Let $U_{nk}, V_{nk}, I_{nk}, J_k$ be as in the proof of Theorem 1. From this proof, it is sufficient to show $J_2 < \infty$ and $J_j < \infty, j = 4,5,6$ with $h(x) = 1$.

For J_2, we first prove that

$$n^{-1/p} \left| \sum_{i=1}^{k_n} a_{ni} EV_{ni} \right| \to 0 \text{ as } n \to \infty.$$

Since $E|X|^{\gamma} \log|X| < \infty$, we have $E|X|^{\gamma} < \infty$ and hence

$$n^{-1/p} \left| \sum_{i=1}^{k_n} a_{ni} EV_{ni} \right| \leq n^{-1/p} \sum_{i=1}^{\infty} n^{-(\gamma-1)/p} E|a_{ni}X_{ni}|^{\gamma} I(|a_{ni}X_{ni}| > n^{1/p})$$

$$\leq n^{-\gamma/p+\beta} E|X|^{\gamma} I(X| > n^{1/p})$$

$$= n^{-(t+1))} E|X|^{\gamma} I(X| > n^{1/p}) \to 0 \text{ as } n \to \infty.$$

Therefore, there exists n large enough such that

$$n^{-1/p} \left| \sum_{i=1}^{k_n} a_{ni} EV_{ni} \right| < \varepsilon/8.$$

Thus, similar to the proof of (15)

$$J_2 \leq C + \sum_{j=0}^{\infty} 2^{j(t+1)} \max_{2^j \leq n < 2^{j+1}} \sum_{i=1}^{k_n} P(|a_{ni}X_{ni}| > n^{1/p})$$

$$\leq C + \sum_{j=0}^{\infty} 2^{j(t+1)} \max_{2^j \leq n < 2^{j+1}} \sum_{i=1}^{\infty} n^{-\gamma/p} E|a_{ni}X_{ni}|^{\gamma} I(|a_{ni}X_{ni}| > n^{1/p})$$

$$\leq C + C\sum_{j=0}^{\infty} 2^{j(t+1)} 2^{-j\gamma/p+j\beta} E|X|^{\gamma} I(|X| > 2^{j/p})$$

$$= C + C\sum_{j=0}^{\infty} E|X|^{\gamma} I(|X| > 2^{j/p})$$

$$= C + C\sum_{j=0}^{\infty}\sum_{i=j}^{\infty} E|X|^{\gamma} I(2^{i/p} < |X| \le 2^{(i+1)/p})$$

$$= C + C\sum_{i=0}^{\infty} iE|X|^{\gamma} I(2^{i/p} < |X| \le 2^{(i+1)/p})$$

$$\le C + CE|X|^{\gamma} \log|X| < \infty.$$

Since $v > \gamma$, we have

$$n^{\beta} = \sum_{i=1}^{\infty} |a_{ni}|^{\gamma} = \sum_{k=1}^{\infty} \sum_{i \in I_{nk}} |a_{ni}|^{\gamma} \ge \sum_{k=1}^{\infty} (\# I_{nk})(k+1)^{-\gamma/p}$$

$$\ge \sum_{k=j}^{\infty} (\# I_{nk})(k+1)^{-v/p} (j+1)^{(v-\gamma)/p}$$

$$> 2^{-v/p} \sum_{k=j}^{\infty} (\# I_{nk}) k^{-v/p} j^{(v-\gamma)/p}.$$

Hence

$$\sum_{k=j}^{\infty} (\# I_{nk}) k^{-v/p} \le Cn^{\beta} j^{-(v-\gamma)/p} \quad \text{for all} \quad j \ge 1. \tag{25}$$

By (25), similar to the proof of (19), we obtain

$$J_5 \le C\sum_{j=0}^{\infty} 2^{j(t+1)} \max_{2^j \le n < 2^{j+1}} n^{-v/p+\beta} \sum_{i=1}^{2n} E|X|^{v} I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$\le C + C\sum_{i=5}^{\infty} i^{(\gamma-v)/p} E|X|^{v} I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$\leq C + CE|X|^{\gamma} < \infty.$$

By (25), similar to the proof of (20), we obtain

$$J_6 \leq C\sum_{i=1}^{\infty} i^{-(v-\gamma)/p} E|X|^v I((i-1)^{1/p} \leq |X| < i^{1/p})$$

$$+ C\sum_{j=1}^{\infty} 2^{j(t+1)} \sum_{i=2^j}^{\infty} 2^{j(\beta-\gamma/p)} i^{-(v-\alpha)/p} E|X|^v I((i-1)^{1/p} \leq |X| < i^{1/p})$$

$$\leq CE|X|^{\gamma} + C\sum_{i=1}^{\infty} i^{-(v-\gamma)/p} E|X|^v I((i-1)^{1/p} \leq |X| < i^{1/p})$$

$$\leq C + C\sum_{i=2}^{\infty} i^{-(v-\gamma)/p} E|X|^v I((i-1)^{1/p} \leq |X| < i^{1/p})$$

$$\leq C + CE|X|^{\gamma} < \infty.$$

Similar to the proof of Theorem 1, we have $J_4 < \infty$.

Remark 2. Obviously, Theorem B follows from Theorem 2 by let $h(x) = 1, X_{ni} = X_i$, for all $i \geq 1, n \geq 1$, and $\{X_i, i \geq 1\}$ be a sequence of independent random variables. Furthermore, Theorem 2 extends and improves the sufficiency part of Theorem C (II) for the case of $\tilde{\rho}$ -mixing random variables.

Corollary 1. *Let* $\{X_{nk}, k \geq 1, n \geq 1\}$ *be an array of zero-mean rowwise* $\tilde{\rho}$ *-mixing random variables stochastically dominated by a random variable* X *. Assume that* $\overline{\lim}_{k\to\infty} \sup_n \tilde{\rho}_n(k) < 1$ *and* $E|X|^p < \infty$ *for some* $p > 2$ *. Let* $\{a_{ni}, i \geq 1, n \geq 1\}$ *be an array of real numbers satisfying (2) and*

$$\sum_{i=1}^{\infty} |a_{ni}|^{\alpha} = O(1) \text{ for some } 2 \leq \alpha < p.$$

Then

$$\sum_{j=0}^{\infty} 2^j \max_{2^j \le n < 2^{j+1}} P\left(n^{-1/p} \left| \sum_{i=1}^{\infty} a_{ni} X_{ni} \right| > \varepsilon \right) < \infty \text{ for all } \varepsilon > 0$$

and

$$\sum_{n=1}^{\infty} P(n^{-1/p} \left| \sum_{i=1}^{\infty} a_{ni} X_{ni} \right| > \varepsilon) < \infty \text{ for all } \varepsilon > 0.$$

Proof. Let $t = 0$ and $\beta = 0$ and $h(x) = 1$. Clearly $|a_{ni}| = O(1)$. Thus the result follows from Theorem 1 (iii).

Corollary 2. *Let* $\{X_{nk}, k \ge 1, n \ge 1\}$ *be an array of zero-mean rowwise* $\tilde{\rho}$ *-mixing random variables stochastically dominated by a random variable* X *. Assume that* $\overline{\lim}_{k \to \infty} \sup_n \tilde{\rho}_n(k) < 1$ *and* $E|X|^2 \log|X| < \infty$. *Let* $\{a_{ni}, i \ge 1, n \ge 1\}$ *be an array of real numbers satisfying*

$$\sum_{i=1}^{\infty} |a_{ni}|^2 = O(1)$$

Then

$$\sum_{j=0}^{\infty} 2^j \max_{2^j \le n < 2^{j+1}} P\left(n^{-1/p} \left| \sum_{i=1}^{\infty} a_{ni} X_{ni} \right| > \varepsilon \right) < \infty \text{ for all } \varepsilon > 0$$

and

$$\sum_{n=1}^{\infty} P(n^{-1/2} \left| \sum_{i=1}^{\infty} a_{ni} X_{ni} \right| > \varepsilon) < \infty \text{ for all } \varepsilon > 0.$$

Proof. Let $t = 0, \beta = 0$, and $p = 2$. Clearly $|a_{ni}| = O(1)$. Thus the result follows from Theorem 2 (ii).

Remark 3. Set $X_{ni} = X_i$ for all $n \geq 1$ and $i \geq 1$, let $\{X_i, i \geq 1\}$ be a sequence of i.i.d. random variables. In this particular case Corollaries 1 and 2 were proved by Li et al. [19]. Hence Corollaries 1 and 2 extend the results of Li et al. [19].

As a corollary of Theorem 1, we can obtain the following result on the rate of convergence for moving average processes.

Corollary 3. *Let* $\{X_{nk}, k \in Z, n \in Z\}$ *be an array of zero-mean rowwise* $\tilde{\rho}$ *-mixing random variables stochastically dominated by a random variable* X *. Assume that* $\overline{\lim}_{k \to \infty} \sup_n \tilde{\rho}_n(k) < 1$ *and* $E|X|^{p(t+2)} < \infty$ *for some* $0 < p < 2$ *and* $p(t+2) > 1$ *. Let* $\{a_n, -\infty < n < \infty\}$ *be a sequence of real numbers such that* $\sum_{n=-\infty}^{\infty} |a_n| < \infty$ *. Set* $a_{ni} = \sum_{j=i+1}^{i+n} a_j$ *for each* i *and* n *. Then*

$$\sum_{j=0}^{\infty} 2^{j(t+1)} \max_{2^j \leq n < 2^{j+1}} P\left(n^{-1/p} \left| \sum_{i=-\infty}^{\infty} a_{ni} X_{ni} \right| > \varepsilon \right) < \infty \text{ for all } \varepsilon > 0,$$

and

$$\sum_{n=1}^{\infty} n^t P\left(\left| \sum_{i=-\infty}^{\infty} a_{ni} X_{ni} \right| / n^{1/p} > \varepsilon\right) < \infty \text{ for all } \varepsilon > 0.$$

Proof. Repeats the proof of Sung [1] and hence omitted.

Remark 4. Corollary 3 extends Corollary 3 of Sung [2] for arrays of rowwise $\tilde{\rho}$ -mixing random variables.

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